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A derivation of the Green-Naghdi equations for irrotational flows

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Abstract. A new derivation of the Green-Naghdi (GN) equations for 'sheet-like' flows is made by use of the principle of virtual work. Divergence-free virtual displacements are used to formulate the momentum equations weakly. This results in the elimination of the internal pressure from the GN equations. As is well-known in particle dynamics, the principle of virtual work can be integrated to obtain Hamilton's principle. These integrations can be performed in a straightforward manner when the Lagrangian description of fluid motion is adopted. When Hamilton's principle is written in an Eulerian reference frame, terms must be added to the Lagrangian to impose the Lin constraint to account for the difference between the Lagrangian and Eulerian variables (Lin). If, however, the Lin constraint is omitted, the scope of Hamilton's principle is confined to irrotational flows (Bretherton). This restricted Hamilton's principle is used to derive the new GN equations for irrotational flows with the same kinematic approximation as in the original derivation of the GN equations. The resulting new hierarchy of governing equations for irrotational flows (referred to herein as the IGN equations) has a considerably simpler structure than the corresponding hierarchy of the original GN governing equations that were not limited to irrotational flows. Finally, it will be shown that the conservation of both the in-sheet and cross-sheet circulation is satisfied more strongly by the IGN equations than by the original GN equations.

Key words: IGN equations, irrotational flow, Hamilton's principle.

1. Introduction

Since the discovery of solitary waves by Russell [1] there has been an increasing number of theories each using different approaches to predict the evolution of long nonlinear waves. Long waves can be considered a subset of more general sheet-like or 'thin' flows where one characteristic dimension of the physical problem is considerably smaller than the other two in a three-dimensional space. There are basically three major approaches to the derivation of long-wave equations.

In the classical approach, conservation of mass and conservation of linear momentum equations form the equations of motion for all fluid particles throughout the continuum. These equations (Navier–Stokes) are approximated by use of several basic assumptions such as the inviscid-fluid and irrotational-flow assumptions. At this point two major scales, namely the dominant-length scale and the dominant-amplitude scale, are introduced. These scales or perturbation parameters are then introduced into approximate equations of motion and into the solid- and free-surface boundary conditions. The velocities and the free-surface elevation, both being unknown, are expanded into a perturbation series ordered in terms of the above scales. Then, one decides which terms in this expansion are retained and which ones are discarded, depending on various factors such as the importance of nonlinearity and dispersion, and even factors such as computational difficulty. The ordered terms collected in the equations of motion and the boundary conditions then result in the desired governing equations for long waves. We note, however, that there are slight variations to this approach, such as the introduction of a depth-averaging process at the outset for the horizontal components of the particle velocity vector. Several well-known shallow-water wave equations, such as the ones by Boussinesq [2] and Korteweg and de Vries [3], and several of their variations follow this classical approach.

Another approach to deriving the long-wave equations is by use of a variational principle, particularly Luke's [4] variational principle. In this method, Euler's Integral is used as the Lagrangian density and several classical equations may be obtained in this way (see Whitham [5]).

In the early 1970s, a new approach to the development of nonlinear long-wave equations was established by A.E. Green of the University of Oxford and Paul M. Naghdi of the University of California at Berkeley. This approach has its roots in the theory of shells and plates in structural mechanics, and is based on a three-dimensional continuum model called a *Cosserat* surface. A *Cosserat* or directed surface is a surface embedded in a three-dimensional Eucledian space with fields of deformable vectors, called 'directors', assigned to every point of this surface. The two-dimensional surface along with the $K \geq 1$ directors model the body of fluid as a deformable medium in a three-dimensional space. In this theory, called the theory of directed fluid sheets [7], the conservation of mass, linear momentum and director momentum are postulated. The essential relations between the stresses and strains given by the constitutive equations are obtained in the form of inertia coefficients for the two-dimensional *Cosserat* surfaces by referring to the three-dimensional equations which are exact. This approach, called the direct approach, then results in a set of nonlinear partial differential equations for the continuum being studied. The general theory does not make the assumption that the fluid is inviscid or that the flow is irrotational, but the fluid is assumed to be incompressible (see, *e.g.*, [8]).

The same system of resulting governing equations for wave propagation in water of variable depth, which were named 'the Green-Naghdi equations' by Ertekin [9], may also be derived from the three-dimensional equations of fluid dynamics in a number of different, but closely related ways. One of the ways is to begin the derivation with the continuity and conservation-of-energy equations in a three-dimensional theory as was done by Green and Naghdi [10]. Then these two statements are specialized to a fluid sheet whose attached directors always remain parallel to the vertical coordinate (this is called the 'restricted' theory) at Level $K = 1$. This assumption, being the only one with regard to the kinematics of the flow field, is certainly a special case of the more general theory of directed fluid sheets. The single director used, in fact, is equivalent to assuming that the horizontal components of the fluid particle velocity are constant and the vertical component varies linearly across the fluid sheet. Then the exact free-surface and solid-boundary conditions are imposed on the integrated conservation equations to obtain a set of nonlinear and time-dependent partial differential equations which also allow a time-dependent sea-floor boundary.

More recently, Shields [11] and Shields and Webster [12] have derived the same equations given by Green and Naghdi [8] by following a different point of view. This set of equations, valid for any number of directors, is obtained from the three-dimensional equations of motion by use of the Kantorovich [13, Chapter IV] method which prescribes the form of the kinematics of the flow in one coordinate direction (here the vertical coordinate), while allowing the solution to vary freely in the other two directions (here the horizontal coordinates). A similar method has been used by Levich and Krylov [14] in their derivation of a set of film-flow equations.

It has been observed that there are similarities between the development of the finite-element method and the Green-Naghdi equations. Kim and Bai [15], for example, found that the finite-element method, based on Hamilton's principle and written in terms of the stream function, gives results very similar to the ones obtained by the GN equations. When they used one linear element in the depthwise direction, they could obtain the same analytical solitary-wave solution produced by the Level I GN equations. Later, Bai and Kim [16] showed that their finite-element solution produces the same dispersion relation that satisfies the GN equations if the same kinematic approximations are used and the wave amplitude is very small, *i.e.*, if the equations are linearized.

Regardless of what approach is taken within the context of the theory of directed fluid sheets described above, one arrives at equations that satisfy the exact boundary conditions and the exact integrated conservation laws. Furthermore, the equations obtained are invariant under a constant superposed rigid-body translation of the whole fluid, *i.e.* they are Galilean invariant, as shown by Green and Naghdi [17].

Part of the present work is an extension to any Level K of Miles and Salmon's [18] work, who derived the Level I GN equations from Hamilton's principle. They showed that both the Lagrangian and Eulerian descriptions result in the Level I GN equations. In the higher-level approximation of the GN equations, Hamilton's principle via the Lagrangian description may not be so useful, since it is difficult to convert the results to Eulerian variables. Instead, we introduce the principle of virtual work which is equivalent to Hamilton's principle for an inviscid fluid. An advantage of using the principle of virtual work is that it can easily be extended to viscous flows, something that is not possible by use of Hamilton's principle. On the other hand, when we use Hamilton's principle in the Eulerian description, we restrict the scope of the principle to irrotational flows by omitting the Lin constraint terms (see Lin [23]), following Miles and Salmon [18].

When we apply the above mentioned variational principles to a sheet of inviscid fluid, we obtain the GN and IGN equations. The principle of virtual work provides the GN equations of any Level K. In this principle, the internal pressure is eliminated through the weak formulation, which employs the divergence-free virtual displacements. Hamilton's principle applied in an Eulerian description gives a new hierarchy of approximate equations, the IGN equations, which have a considerably simpler structure than the original GN equations.

The derivations of the variational principle and the corresponding approximate equations are achieved by use of the divergence-free velocity and displacement fields to satisfy the continuity equation *a priori*. The divergence-free field, say $\mathbf{v}$, can be given in terms of a vector stream function $\boldsymbol{\Psi}(x^1, x^2, x^3, t) = (\Psi^1, \Psi^2, \Psi^3)$ as

$$\mathbf{v} = \nabla \times \boldsymbol{\Psi}.$$

Without loss of generality for sheet-like flows, we can omit the third component, Ψ^3, of $\boldsymbol{\Psi}$ by setting it equal to zero. As a result, the variational formulations are given in terms of $\boldsymbol{\Psi}$, which has one less component than the original vector function $\mathbf{v}$. As a consequence of using these divergence-free velocity fields, the results of the variational principles are slightly different from Euler's equations which have been the starting point of former uses of variational principles.

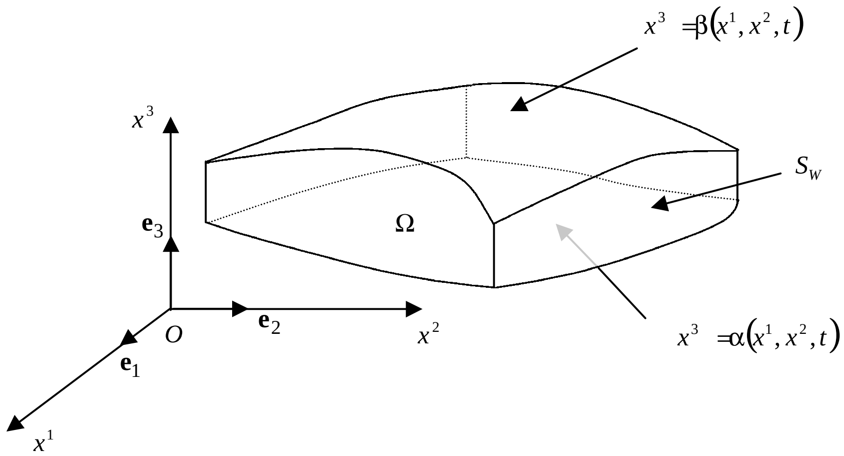

Figure 1. Definition sketch.

The principle of virtual work and Hamilton's principle in a Lagrangian description result equivalently in the vorticity-transport equations. In the case of Hamilton's principle in an Eulerian reference frame, we can obtain the irrotationality condition: $\nabla \times \mathbf{v} = 0$. The vorticity-transport equations and the irrotationality equations are weakly satisfied in the approximate equations. The weak conservation of circulation obtained by Shields and Webster [19] is derived directly from approximate vorticity-transport equations.

2. Statement of the problem

In the following, we shall employ the standard Cartesian-tensor notation, with the summation convention implied for repeated indices. Latin indices indicate quantities having three spatial components and they take the values 1, 2 or 3; Greek indices take the values 1 or 2 only.

Let (x^1, x^2, x^3) be a right-handed system of fixed, rectangular Cartesian coordinates with base vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, where positive $\mathbf{e}_3$ is oriented vertically upwards. With reference to Figure 1, we consider the motion of a sheet-like body of incompressible and inviscid fluid in a gravitational field $-g\mathbf{e}_3$. The fluid domain Ω is assumed to be bounded from above and below by two smooth, non-intersecting, time-varying material surfaces:

$$x^3 = \alpha(x^1, x^2, t), \quad x^3 = \beta(x^1, x^2, t), \quad \beta > \alpha. \tag{2.1}$$

The local thickness η and the mid-surface location ζ can be defined by

$$\eta(x^1, x^2, t) = \beta - \alpha, \quad \zeta(x^1, x^2, t) = \tfrac{1}{2}(\beta + \alpha). \tag{2.2}$$

The equations of motion for the inviscid-fluid body are the continuity equation and Euler's equations:

$$v^i_{,i} = 0, \tag{2.3}$$

$$\mathbf{v}_{,t} + v^i \mathbf{v}_{,i} = -\frac{1}{\rho} p_{,i} \mathbf{e}_i - g\mathbf{e}_3, \tag{2.4}$$

respectively, where $\mathbf{v} = v^i(x^1, x^2, x^3, t)\mathbf{e}_i$ and $p(x^1, x^2, x^3, t)$ are the velocity vector and pressure defined in the fluid domain Ω, respectively.

In the absence of surface tension, the principal of equivalence of external and internal pressures on the surfaces α and β yields the dynamic boundary conditions on these surfaces:

$$p|_{x^3=\beta} = \hat{p}, \quad p|_{x^3=\alpha} = \bar{p}, \tag{2.5}$$

where $\hat{p}$ and $\bar{p}$ are the pressures acting on the upper and lower surfaces of the fluid sheet, respectively. Hereafter, we shall use the notation ˆ (hat) and ¯ (bar) over a variable to represent the values of a function evaluated on $x^3 = \beta$ and on $x^3 = \alpha$, respectively. The kinematic boundary conditions follow from the hypothesis that α and β are material surfaces, which imposes the constraint that the normal velocities of the particles on the surfaces are identical to those of the normal velocities of the surfaces themselves. Thus,

$$\left[v^3 - \beta_{,t} - v^\gamma \beta_{,\gamma}\right]_{x^3=\beta} = 0, \tag{2.6}$$

$$\left[v^3 - \alpha_{,t} - v^\gamma \alpha_{,\gamma}\right]_{x^3=\alpha} = 0. \tag{2.7}$$

We complete the problem statement by enforcing the additional boundary conditions on the vertical control surface S_W, closing the domain (see Figure 1). We assume that the normal velocity is specified on this vertical control surface.

3. GN equations (direct method)

Shields and Webster [12] derived the GN equations following the variational method of Kantorovich (also referred to as 'direct method') (see, for example, Kantorovich and Krylov [13, Chapter IV]). In their derivation, they assumed an approximate velocity field in the vertical direction while satisfying the continuity equation and the kinematic boundary conditions exactly; on the other hand, Euler's equations were weakly formulated. We recall their results for later comparison with the developments in this paper.

For convenience, a transformed coordinate s is adopted:

$$s(x^1, x^2, x^3, t) = \frac{2}{\eta}\left(x^3 - \zeta\right). \tag{3.1}$$

As defined, the function s maps the fluid domain to a region between two parallel planes in the coordinate system (x^1, x^2, s), given by $|s| \leq 1$. In particular, the upper surface of the fluid is mapped to $s = +1$, the lower surface is mapped to $s = -1$, and the midsurface is mapped to $s = 0$. The velocity field is assumed to have the following polynomial expansion of order K:

$$\mathbf{v}(x^1, x^2, x^3, t) = \sum_{n=0}^{K} \mathbf{w}_n(x^1, x^2, t)s^n, \tag{3.2}$$

where

$$\mathbf{w}_n(x^1, x^2, t) = w_n^i(x^1, x^2, t)\mathbf{e}_i, \quad n = 0, 1, \ldots, K-1, \tag{3.3a}$$

$$\mathbf{w}_K(x^1, x^2, t) = w_K^3(x^1, x^2, t)\mathbf{e}_3. \tag{3.3b}$$

Then the continuity equation (2.3) and the kinematic boundary conditions (2.6) and (2.7) are given as follows.

Continuity equation:

$$\zeta_{,\gamma} w_n^{\gamma} + \frac{1}{2}\eta_{,\gamma} w_{n-1}^{\gamma} - w_n^3 = \frac{1}{2n}\left(\eta w_{n-1}^{\gamma}\right)_{,\gamma}, \quad n = 1, 2, \ldots, K. \tag{3.4}$$

Kinematic boundary conditions:

$$2\sum_{n=1,3,\ldots}^{K} w_n^3 = \eta_{,t} + \eta_{,\gamma}\sum_{n=0,2,\ldots}^{K-1} w_n^{\gamma} + 2\zeta_{,\gamma}\sum_{n=1,3,\ldots}^{K-1} w_n^{\gamma}, \tag{3.5}$$

and

$$2\sum_{n=0,2,\ldots}^{K} w_n^3 = 2\zeta_{,t} + \eta_{,\gamma}\sum_{n=1,3,\ldots}^{K-1} w_n^{\gamma} + 2\zeta_{,\gamma}\sum_{n=0,2,\ldots}^{K-1} w_n^{\gamma}, \tag{3.6}$$

where the summations are over even or odd indices, as indicated.

It should be noted that the above kinematic constraints are exactly satisfied as given. We satisfy the momentum equations (2.4) weakly by use of the powers of s as the test functions. To do that, one can integrate (2.4) over s, after multiplying it by s^m, $m = 0, 1, \ldots, K-1$:

$$\begin{aligned}
\frac{\eta}{2}\int_{-1}^{1} s^m \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\,\mathrm{d}s &= \sum_{n=0}^{K}\Big[\eta\theta_{m+n}\left(\mathbf{w}_{n,t} + w_0^{\gamma}\mathbf{w}_{n,\gamma}\right) \\
&+ \sum_{r=1}^{K-1}\theta_{m+n+r}\left\{\eta w_r^{\gamma}\mathbf{w}_{n,\gamma} + n\mu_{m+n}^{r}\left(\eta w_r^{\gamma}\right)_{,\lambda}\mathbf{w}_n\right\}\Big] \\
&= -\frac{1}{\rho}\left\{p_{m,\gamma} + mp_m\frac{\eta_{,\gamma}}{\eta} - 2mp_{m-1}\frac{\zeta_{,\gamma}}{\eta} - \hat{p}\beta_{,\gamma} + (-1)^m\bar{p}\alpha_{,\gamma}\right\}\mathbf{e}_{\gamma} \\
&- \frac{1}{\rho}\left\{\hat{p} - (-1)^m\bar{p} - \frac{2mp_{m-1}}{\eta} + \rho g\eta\theta_m\right\}\mathbf{e}_3,
\end{aligned} \tag{3.7}$$

and one can integrate the vertical component of (2.4) over s, after multiplying it by s^K:

$$\begin{aligned}
\frac{\eta}{2}\int_{-1}^{1} s^K \frac{\mathrm{D}v^3}{\mathrm{D}t}\mathrm{d}s &= \sum_{n=0}^{K}\Big[\eta\theta_{K+n}\left(w_{n,t}^3 + w_0^{\gamma}w_{n,\gamma}^3\right) + \\
&+ \sum_{r=1}^{K-1}\theta_{K+n+r}\left\{\eta w_r^{\gamma}w_{n,\gamma}^3 + n\mu_{K+n}^{r}\left(\eta w_r^{\gamma}\right)_{,\gamma}w_n^3\right\}\Big] \\
&= -\frac{1}{\rho}\left\{\hat{p} - (-1)^K\bar{p} - \frac{2Kp_{K-1}}{\eta} + \rho g\eta\theta_K\right\}\mathbf{e}_3,
\end{aligned} \tag{3.8}$$

where

$$\frac{\mathrm{D}\cdot}{\mathrm{D}t} = \frac{\partial\cdot}{\partial t} + v^i\frac{\partial\cdot}{\partial x^i}$$

is defined as the material derivative, and

$$\theta_n = \begin{cases} 1/(n+1), & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \tag{3.9}$$

$$\mu_n^r = \begin{cases} 1/n, & r \text{ odd}, \\ r/\{(r+1)(n+1)\}, & r \text{ even}, \end{cases} \tag{3.10}$$

$$p_n = (\eta/2) \int_{-1}^{1} s^n p \, \mathrm{d}s. \tag{3.11}$$

In the applications of the theory (*e.g.*, [9], [12] and [20]); Shields and Webster [12]), the GN equations are reduced by elimination of the pressure integrals p_n and the vertical velocity components w_n^3. The results can be given as partial differential equations for $\left\{\zeta, \eta, w_n^\gamma, n = 0, \dots, K-1\right\}$. In the past, the reduction of the GN equations has been done separately at each level of approximation and for $K \leq 3$. However, it has not been known that the elimination of the pressure integrals are possible at any level of approximation. We shall show that this is always possible by using the principle of virtual work in the next section.

4. GN equations (principle of virtual work)

4.1. Statement of the principle

In the formulation of the GN equations given by (3.4)–(3.8), the assumed solution (3.2) and the test functions, s^n, are taken without any further restriction. We shall next impose a restriction on the test functions before the weak formulation is made.

We consider the velocity and displacement fields that satisfy the continuity equation in Ω and the vanishing normal virtual displacement field on S_W:

$$v^i_{,i} = 0, \qquad \delta X^i_{,i} = 0, \quad \text{in} \quad \Omega, \tag{4.1a}$$

$$\delta X^i n_i = 0, \qquad \text{on} \quad S_W, \tag{4.1b}$$

where δX^i is the virtual displacement that will be used here as the test function. Equation (4.1b) states that the normal component of the virtual displacement is assumed to be zero on the vertical control surface S_W where the normal velocity is specified. The virtual work done by the virtual displacement $\delta \mathbf{X}(x^1, x^2, x^3, t) = \delta X^i \mathbf{e}_i$ can be written as

$$\delta W = \int_\Omega \delta \mathbf{X} \cdot \left\{\rho \left(\mathbf{v}_{,t} + v^i \mathbf{v}_{,i}\right) + p_{,i}\mathbf{e}_i + \rho g \mathbf{e}_3\right\} \mathrm{d}V = 0. \tag{4.2}$$

The virtual work due to the pressure and gravitational terms can be obtained as

$$\begin{aligned} &\int_\Omega \delta X^i (p + \rho g x^3)_{,i} \mathrm{d}V = \int_{\partial\Omega} \delta X^i n^i (p + \rho g x^3) \, \mathrm{d}S \\ &= \iint_{\partial\Omega} \left(\delta X^3 - \beta_{,\gamma} \delta X^\gamma\right)_{x^3=\beta} (\hat{p} + \rho g \beta) \, \mathrm{d}x^1 \, \mathrm{d}x^2 \\ &\quad - \iint \left(\delta X^3 - \alpha_{,\gamma} \delta X^\gamma\right)_{x^3=\alpha} (\bar{p} + \rho g \alpha) \, \mathrm{d}x^1 \, \mathrm{d}x^2, \end{aligned} \tag{4.3}$$

where $\partial\Omega$ is the entire boundary surface enclosing Ω, and where the divergence theorem has been applied and the dynamic boundary conditions (2.5) have been enforced. Note that the virtual work due to the pressure has contributions only from the top and bottom boundaries.

If we now write $\delta W = \iint \delta w \, \mathrm{d}x^1 \, \mathrm{d}x^2$, the virtual-work density (per unit area), δw, can be written as

$$\delta w = \rho \int_\alpha^\beta \delta \mathbf{X} \cdot \left(\mathbf{v}_{,t} + v^i \mathbf{v}_{,i}\right) \mathrm{d}x^3 + \left(\delta X^3 - \beta_{,\gamma} \delta X^\gamma\right)_{x^3=\beta} (\hat{p} + \rho g \beta) \\ - \left(\delta X^3 - \alpha_{,\gamma} \delta X^\gamma\right)_{x^3=\alpha} (\bar{p} + \rho g \alpha), \tag{4.4}$$

where (4.2) and (4.3) were used. The principle of virtual work states that the above virtual work is equal to zero for any divergence-free virtual displacement that satisfies (4.1).

4.2. Depth-integrated momentum equations

We introduce a stream function, $\delta\boldsymbol{\Psi}$, to define the divergence-free virtual displacement as

$$\delta \mathbf{X} = \boldsymbol{\nabla} \times \delta \boldsymbol{\Psi}. \tag{4.5}$$

For a given divergence-free vector field $\delta\mathbf{X}$, the stream function is not unique. We choose a stream function such that it has no vertical component. It is also advantageous to introduce a vector potential, $\delta\boldsymbol{\Phi}(x^1, x^2, x^3, t) = \delta\Phi^\gamma \mathbf{e}_\gamma$, by which the stream function is defined as

$$\delta \boldsymbol{\Psi}(x^1, x^2, x^3, t) = \left(\delta\Phi^2, -\delta\Phi^1, 0\right).$$

The virtual displacement can then be written as

$$\delta X^\gamma = \delta \Phi^\gamma_{,3}, \qquad \delta X^3 = -\delta \Phi^\gamma_{,\gamma}. \tag{4.6}$$

The principle of virtual work in (4.4) can be invoked to derive a depth-integrated momentum equation. Following the procedure of Kantorovich, we approximate the vector potentials using a sequence of interpolation functions, $\{f_n(s), n = 0, 1, \ldots\}$, which is a complete set in the interval $(-1, 1)$. If we truncate the interpolation at $n = K$, we have

$$\delta\Phi^\gamma(x^1, x^2, x^3, t) = f_n(s) \delta \Phi^\gamma_n(x^1, x^2, t), \quad n = 0, 1, \ldots, K, \tag{4.7}$$

and the virtual displacements can be written as

$$\delta X^\gamma = \frac{2}{\eta} f_n'(s) \delta \Phi^\gamma_n, \tag{4.8a}$$

$$\delta X^3 = -f_n(s) \delta \Phi^\gamma_{n,\gamma} + \frac{1}{\eta} f_n'(s) \delta \Phi^\gamma_n (2\zeta_{,\gamma} + s\eta_{,\gamma}), \tag{4.8b}$$

where the prime denotes differentiation with respect to s. Inserting the virtual displacements given by (4.8*a*) and (4.8*b*) in (4.4), we obtain the virtual work per unit area follows as

$$\delta w = \delta \Phi^\gamma_n \left[\rho \int_{-1}^{1} \left\{ f_n'(s) \frac{\mathrm{D}v^\gamma}{\mathrm{D}t} + f_n'(s) \left(\zeta_{,\gamma} + \frac{s}{2} \eta_{,\gamma}\right) \frac{\mathrm{D}v^3}{\mathrm{D}t} \right\} \mathrm{d}s + \\ + \rho \left(\int_{-1}^{1} f_n(s) \frac{\eta}{2} \frac{\mathrm{D}v^3}{\mathrm{D}t} \, \mathrm{d}s \right)_{,\gamma} + f_n(1)(\hat{p} + \rho g \beta)_{,\gamma} - f_n(-1)(\bar{p} + \rho g \alpha)_{,\gamma} \right]. \tag{4.9}$$

In obtaining (4.9), we used the following identity:

$$\iint \delta\Phi^{\gamma}_{n,\gamma}(\cdot)\,\mathrm{d}x^1\,\mathrm{d}x^2 = \oint_{C_W} \delta\Phi^{\gamma}_{n} n^{\gamma}(\cdot)\,\mathrm{d}l - \iint \delta\Phi^{\gamma}_{n}(\cdot)_{,\gamma}\,\mathrm{d}x^1\,\mathrm{d}x^2$$

$$= -\iint \delta\Phi^{\gamma}_{n}(\cdot)_{,\gamma}\,\mathrm{d}x^1\,\mathrm{d}x^2$$

for an arbitrary scalar variable $(\cdot)$. The line integral is equal to zero since $\delta\Phi^{\gamma}_{n} n^{\gamma} = 0$ on S_W, which can be shown from (4.1b) and (4.8a). The contour C_W denotes the projection of the vertical control surface S_W onto the Ox^1x^2 plane. Since the functions $\delta\Phi^{\gamma}_{n}(x^1, x^2, t)$ are arbitrary, the principle of virtual work results in the following depth-integrated momentum equations:

$$\rho\int_{-1}^{1}\left\{f_n'(s)\frac{\mathrm{D}v^{\gamma}}{\mathrm{D}t} + f_n'(s)\left(\zeta_{,\gamma} + \frac{s}{2}\eta_{,\gamma}\right)\frac{\mathrm{D}v^3}{\mathrm{D}t}\right\}\mathrm{d}s + \rho\left(\frac{\eta}{2}\int_{-1}^{1} f_n(s)\frac{\mathrm{D}v^3}{\mathrm{D}t}\,\mathrm{d}s\right)_{,\gamma}$$
$$= -f_n(1)\left(\hat{p} + \rho g\beta\right)_{,\gamma} + f_n(-1)\left(\bar{p} + \rho g\alpha\right)_{,\gamma},\; n = 0, 1, \ldots, K. \tag{4.10}$$

If we particularly choose $f_n(s)$ as the same polynomial functions used by Shields and Webster [12], *i.e.*,

$$f_n(s) = s^n, \quad n = 0, 1, \ldots, K,$$

(4.9) becomes

$$n\int_{-1}^{1} s^{n-1}\frac{\mathrm{D}v^{\gamma}}{\mathrm{D}t}\,\mathrm{d}s + n\int_{-1}^{1} s^{n-1}\left(\zeta_{,\gamma} + \frac{s}{2}\eta_{,\gamma}\right)\frac{\mathrm{D}v^3}{\mathrm{D}t}\,\mathrm{d}s + \left(\frac{\eta}{2}\int_{-1}^{1} s^{n}\frac{\mathrm{D}v^3}{\mathrm{D}t}\,\mathrm{d}s\right)_{,\gamma}$$
$$= -\frac{1}{\rho}\left\{\left(\hat{p} + \rho g\beta\right)_{,\gamma} - (-1)^n\left(\bar{p} + \rho g\alpha\right)_{,\gamma}\right\},\; n = 0, 1, \ldots, K. \tag{4.11}$$

Equation (4.11) is obtainable by eliminating the p_n variables from the original GN equations given by (3.7) and (3.8).

At first glance, this formulation looks more complicated than the one given in Shields and Webster [19]; however, there is a real difference: the p_n variables, (3.11), no longer exist.

4.3. Derivation of the Level IGN equations

Now we consider the free-surface motion due to a given bottom motion $\alpha(x^1, x^2, t)$ and surface pressure forcing $\hat{p}(x^1, x^2, t)$. To derive the Level I GN equations, we assume that the horizontal velocity components are independent of the vertical coordinate x^3 and that the vertical component varies linearly along the vertical x^3-axis. If we define $\mathbf{u} = u^{\gamma}(x^1, x^2, t)\mathbf{e}_{\gamma}$ as the depth-independent horizontal-velocity field, the approximate velocity field, which satisfies the continuity equation and the bottom boundary condition, given by (2.3) and (2.7), respectively, can be written as

$$v^{\gamma} = u^{\gamma}(x^1, x^2, t), \quad v^3 = -\frac{\eta}{2}(1+s)\nabla\cdot\mathbf{u} + \mathcal{D}\alpha, \tag{4.12}$$

where we have defined $\mathcal{D} \equiv (\partial/\partial t) + \mathbf{u}\cdot\nabla$ as the material derivative in the horizontal plane. Substituting (4.12) in the kinematic boundary condition, (2.6), on the top surface of the sheet, we have

$$\nabla \cdot \mathbf{u} = -\frac{\mathcal{D}\eta}{\eta}. \tag{4.13}$$

Then we can write

$$v^3 = \mathcal{D}\zeta + \frac{s}{2}\mathcal{D}\eta,$$

and the horizontal and vertical accelerations can be obtained as

$$\frac{\mathrm{D}v^\gamma}{\mathrm{D}t} = \mathcal{D}u^\gamma, \frac{\mathrm{D}v^3}{\mathrm{D}t} = \mathcal{D}^2\zeta + \tfrac{s}{2}\mathcal{D}^2\eta. \tag{4.14}$$

Substituting (4.14) in (4.11) and adding the two equations for $n = 0$ and $n = 1$, we obtain

$$\mathcal{D}\mathbf{u} + \nabla\left\{\frac{\eta}{2}\left(\mathcal{D}^2\zeta + \frac{1}{6}\mathcal{D}^2\eta\right)\right\} + \nabla\zeta\,\mathcal{D}^2\zeta + \frac{1}{12}\nabla\eta\mathcal{D}^2\eta = -g\nabla\beta - \frac{\nabla\hat{p}}{\rho}, \tag{4.15}$$

which is the same equation derived by Miles and Salmon [18] who used Hamilton's principle with a Lagrangian description. We can also obtain

$$\bar{p} = \rho\eta\left(\mathcal{D}^2\zeta + g\right) + \hat{p}, \tag{4.16}$$

after substituting (4.14) in (4.11) with $n = 0$, *i.e.*, $f_0(s) = 1$.

Substituting the relations given in (2.2) in (4.15) and (4.16), we obtain

$$\mathcal{D}\mathbf{u} + \frac{1}{6}\left\{\nabla(2\beta+\alpha)\mathcal{D}^2\alpha + \nabla(4\beta-\alpha)\mathcal{D}^2\beta + \eta\nabla(2\mathcal{D}^2\beta + \mathcal{D}^2\alpha)\right\} = -g\nabla\beta - \frac{\nabla\hat{p}}{\rho},$$

and

$$\bar{p} = \frac{\rho\eta}{2}\left(\mathcal{D}^2\beta + \mathcal{D}^2\alpha + 2g\right) + \hat{p},$$

which are identical to (2.22)–(2.23) and (2.21), respectively, derived by Ertekin [9].

4.4. Higher-level equations

The full hierarchy of approximate equations can be obtained if we represent the divergence-free velocity vector $\mathbf{v}(x^1, x^2, x^3, t)$ by a vector potential $\mathbf{\Phi} = (\Phi^1, \Phi^2, 0)$ such that $v^\gamma = \delta\Phi^\gamma_{,3}$, $v^3 = -\delta\Phi^\gamma_{,\gamma}$, as we did for the virtual displacement in (4.6), and expand $\mathbf{\Phi}$ using the same interpolation functions used in (4.7):

$$\Phi^\gamma(x^1, x^2, x^3, t) = f_n(s)\Phi^\gamma_n(x^1, x^2, x^3, t), \quad n = 0, 1, \ldots, K, \tag{4.17a}$$

$$v^\gamma = \frac{2}{\eta} f_n'(s)\Phi^\gamma_n, \tag{4.17b}$$

$$v^3 = -f_n(s)\Phi^\gamma_{n,\gamma} + \frac{1}{\eta} f_n'(s)\Phi^\gamma_n(2\zeta_{,\gamma} + s\eta_{,\gamma}), \tag{4.17b}$$

where the summation convention is used as usual.

The relation between $\mathbf{w}_n(x^1, x^2, t)$ introduced in (3.2) and the vector potential $\mathbf{\Phi}_n(x^1, x^2, t)$ can be obtained by comparing the expansions of the velocity vector given by (3.2) and (4.17) with $f_n(s) = s^n$:

$$\mathbf{w}_n = \frac{2(n+1)}{\eta}\boldsymbol{\Phi}_{n+1} + \mathbf{e}_3\Big\{ -\nabla\cdot\boldsymbol{\Phi}_n + \frac{2(n+1)}{\eta}\nabla\zeta\cdot\boldsymbol{\Phi}_{n+1} + \frac{n}{\eta}\nabla\eta\cdot\boldsymbol{\Phi}_n\Big\}, \quad n = 0, 1, \ldots, K-1, \tag{4.18}$$

$$\mathbf{w}_K = \left(-\nabla\cdot\boldsymbol{\Phi}_K + \frac{K}{\eta}\nabla\eta\cdot\boldsymbol{\Phi}_K\right)\mathbf{e}_3,$$

which obviously satisfies the continuity equation given by (3.4). If we substitute (4.18) in (3.5) and (3.6), we obtain the kinematic boundary conditions on the top and bottom surfaces, respectively, of the fluid sheet as

$$\eta_t + 2\sum_{n=1,3,5,\ldots}^{K}\nabla\cdot\boldsymbol{\Phi}_n = 0, \quad \zeta_t + 2\sum_{n=0,2,4,\ldots}^{K}\nabla\cdot\boldsymbol{\Phi}_n = 0. \tag{4.19}$$

Note that the first equation of (4.19) can be written as

$$\eta_t + \nabla\cdot\left\{\boldsymbol{\Phi}(x^1, x^2, \beta, t) - \boldsymbol{\Phi}(x^1, x^2, \alpha, t)\right\} = 0,$$

or, from the relation $v^\gamma = \delta\Phi^\gamma_{,3}$,

$$\eta_t + \left\{\int_\alpha^\beta v^\gamma\, \mathrm{d}x^3\right\}_{,\gamma} = 0,$$

which can be interpreted as the statement of mass conservation for a control volume bounded by a vertical surface and the top and bottom surfaces of the fluid sheet.

We can also obtain the momentum equations of the GN equations (4.11) in terms of $\boldsymbol{\Phi}_n(x^1, x^2, t) = (\Phi^1_n, \Phi^2_n, 0)$ by using the polynomial expansion $f_n(s) = s^n$ as follows:

$$\sum_{n=0}^{K}\Bigg[\frac{4mn}{\eta}\theta_{m+n-2}\boldsymbol{\Phi}_{n,t} - m\nabla\left(2\theta_{m+n-1}\zeta + \theta_{m+n}\eta\right)\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_{n,t} - \theta_{m+n}\nabla\left(\eta\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_{n,t}\right)$$
$$+\frac{2mn}{\eta}\nabla\zeta\theta_{m+n-2}\nabla\zeta + \theta_{m+n-1}\nabla\eta)\cdot\boldsymbol{\Phi}_{n,t}$$
$$+\frac{mn}{\eta}\nabla\eta(2\theta_{m+n-2}\nabla\zeta + \theta_{m+n}\nabla\eta)\cdot\boldsymbol{\Phi}_{n,t} + n\nabla\left\{(2\theta_{m+n-1}\nabla\zeta + \theta_{m+n}\nabla\eta)\cdot\boldsymbol{\Phi}_{n,t}\right\}\Bigg]$$
$$+\sum_{n=0}^{K}\Bigg[-\frac{4mn\eta_{,t}}{\eta^2}\theta_{m+n-2}\boldsymbol{\Phi}_n + 2mn\nabla\zeta\left\{2\theta_{m+n-2}\left(\frac{\nabla\zeta}{\eta}\right)_{,t} + \theta_{m+n-1}\left(\frac{\nabla\eta}{\eta}\right)_{,t}\right\}\cdot\boldsymbol{\Phi}_n$$
$$+mn\nabla\eta\left\{2\theta_{m+n-1}\left(\frac{\nabla\zeta}{\eta}\right)_{,t} + \theta_{m+n}\left(\frac{\nabla\eta}{\eta}\right)_{,t}\right\}\cdot\boldsymbol{\Phi}_n$$
$$+n\boldsymbol{\nabla}\left\{2\theta_{m+n-1}\left(\frac{\nabla\zeta}{\eta}\right)_{,t} + \theta_{m+n}\left(\frac{\nabla\eta}{\eta}\right)_{,t}\right\}\cdot\boldsymbol{\Phi}_n\Bigg]$$

$$
\begin{aligned}
&+8m \sum_{n=0}^{K} \left[\frac{n}{\eta} \theta_{m+n-2} \boldsymbol{\Phi}_1 \cdot \nabla \left(\frac{1}{\eta} \boldsymbol{\Phi}_n \right) \right. \\
&\qquad \left. + \sum_{r=2}^{K} \frac{rn}{\eta} \theta_{m+n+r-3} \left\{ \boldsymbol{\Phi}_r \cdot \nabla \left(\frac{1}{\eta} \boldsymbol{\Phi}_n \right) + \frac{n}{\eta} \mu_{m+n-2}^{r-1} \boldsymbol{\nabla} \cdot \boldsymbol{\Phi}_r \boldsymbol{\Phi}_n \right\} \right] \\
&+m \nabla \zeta F_{m-1} + \frac{m}{2} \nabla \eta F_m + \nabla \left(\frac{\eta}{2} F_m \right) \\
&= -\frac{1}{\rho} \left\{ \left(\hat{p} + \rho g \beta \right)_{,\gamma} - (-1)^m \left(\bar{p} + \rho g \alpha \right)_{,\gamma} \right\}, \quad m = 0, 1, \ldots, K,
\end{aligned} \tag{4.20}
$$

where F_m is defined as

$$
\begin{aligned}
F_m = \frac{4}{\eta} \sum_{n=0}^{K} \Bigg[& - \theta_{m+n} \boldsymbol{\Phi}_1 \cdot \nabla \left(\boldsymbol{\nabla} \cdot \boldsymbol{\Phi}_n \right) \\
&+ n \boldsymbol{\Phi}_1 \cdot \nabla \left\{ \frac{1}{\eta} \left(2\theta_{m+n-1} \nabla \zeta + \theta_{m+n} \nabla \eta \right) \cdot \boldsymbol{\Phi}_n \right\} \\
&+ \sum_{r=2}^{K} \Big[- r\theta_{m+n+r-1} \left\{ \boldsymbol{\Phi}_r \cdot \boldsymbol{\nabla} (\nabla \cdot \boldsymbol{\Phi}_n) + n \mu_{m+n}^{r-1} \boldsymbol{\nabla} \cdot \boldsymbol{\Phi}_r \nabla \cdot \boldsymbol{\Phi}_n \right\} \\
&+ rn \boldsymbol{\Phi}_r \cdot \nabla \left\{ \frac{1}{\eta} \left(2\theta_{m+n+r-2} \nabla \zeta + \theta_{m+n+r-1} \nabla \eta \right) \cdot \boldsymbol{\Phi}_n \right\} \\
&+ \frac{rn}{\eta} \boldsymbol{\nabla} \cdot \boldsymbol{\Phi}_r \left\{ n \theta_{m+n+r-1} \mu_{m+n}^{r-1} \nabla \eta \cdot \boldsymbol{\Phi}_n + 2(n-1) \theta_{m+n+r-2} \mu_{m+n-1}^{r-1} \nabla \zeta \cdot \boldsymbol{\Phi}_n \right\} \Big] \Bigg].
\end{aligned} \tag{4.21}
$$

It is very important to note that the pressure integral p_n, (3.11), no longer appears in the new form of the GN equations given by (4.19) and (4.20) and that the θ_n variables are actually known numbers from (3.9). Moreover, the continuity equations, (3.4), of the original GN equations are no longer needed.

We shall show later in Section 6 that the new momentum equations of the GN equations given by (4.20) are actually the depth-integrated vorticity transport equations. As shown by Shields and Webster [19], the GN equations conserve vorticity in a weak sense and thus, they are able to treat inviscid rotational flows. However, if we begin by restricting the flows to irrotational flows, then it is possible to simplify the GN equations given by (4.19) and (4.20). The assumption of irrotationality permits Euler's equations to be integrated to obtain Euler's integral. In the context of the variational approach, the assumption of irrotationality allows the integration of the principle of virtual work to yield Hamilton's principle. This will be discussed next.

5. IGN equations (Hamilton's principle)

It has been known that Euler's equations can be derived from Hamilton's principle in several ways (Herivel [21]; Serrin [22]; Lin [23]; Seliger and Whitham [24]). The derivation can be most directly carried out in a Lagrangian description of the fluid motion. The principle can simply be stated as the stationary condition of the Lagrangian, L, defined as the difference between the kinetic and potential energies. This is the principle of least action. Miles and Salmon [18] used this principle to derive the Level I GN equations. However, the extension of the method to a higher level has not been feasible since the transformation between the Lagrangian and Eulerian descriptions cannot easily be done unless the velocity field is as simple as the columnar motion assumed in the Level I theory. Miles and Salmon [18] also showed that the irrotational version of the Level I GN equations can be derived from Hamilton's principle written in an Eulerian framework. We will follow the latter approach to derive the irrotational version of the GN equations.

The most general form of the Lagrangian given in an Eulerian reference frame can be found in Lin [23]. Additional Lagrange multiplier terms are added to the Lagrangian to enforce the continuity equation and the Lin constraint. The Lin constraint specifies the relation between the Eulerian and Lagrangian variables and allows Hamilton's principle to describe more general flows. If one omits the Lin constraint, the scope of Hamilton's principle is restricted to irrotational flows (Bretherton [25]). We shall exploit this property to derive the irrotational version of the GN equations.

5.1. Hamilton's principle for irrotational flows

Hamilton's principle states that the true path of a particle makes the action integral $\int L \, \mathrm{d}t$ have a stationary value for an arbitrary variation of the particle path (see, *e.g.* Goldstein [26 Chapter 8]). When there is a kinematic constraint on the motion of the particle, we either find the solution among the trial solutions that satisfy the constraint *a priori*, or use a Lagrange multiplier to satisfy the constraints.

The same principle can be applied to a fluid continuum. The Lagrangian L is defined as the difference between the kinetic energy T and potential energy V:

$$T = \frac{\rho}{2} \int_{\Omega} \mathbf{v} \cdot \mathbf{v} \, \mathrm{d}V, \quad V = \rho g \int_{\Omega} x^3 \, \mathrm{d}V = \frac{\rho g}{2} \iint \left(\beta^2 - \alpha^2\right) \mathrm{d}x^1 \, \mathrm{d}x^2. \tag{5.1}$$

When there is an external pressure on a moving boundary, we add the work done by the pressure:

$$W = - \iint \left(\hat{p}\beta - \bar{p}\alpha\right) \, \mathrm{d}x^1 \, \mathrm{d}x^2. \tag{5.2}$$

The continuity equation (2.3) and the kinematic boundary conditions (2.6) and (2.7) are treated as kinematic constraints. The continuity equation is assumed to be satisfied by a divergence-free trial solution $\mathbf{v}$ as in Section 4 and the kinematic boundary conditions are embedded by the Lagrange multipliers. Then the Lagrangian can be written as $L = T - V + W$ plus the Lagrange multiplier terms:

$$\begin{aligned} L = \iint \, \mathrm{d}x^1 \, \mathrm{d}x^2 \Bigg[& \frac{\rho}{2} \int_{\alpha}^{\beta} \mathbf{v} \cdot \mathbf{v} \, \mathrm{d}x^3 - \frac{\rho g}{2} (\beta^2 - \alpha^2) - (\hat{p}\beta - \bar{p}\alpha) \\ & + \rho \hat{\phi} (\beta_{,t} - \hat{v}^3 + \beta_{,t} \hat{v}^{\gamma}) - \rho \bar{\phi} (\alpha_{,t} - \bar{v}^3 + \alpha_{,t} \bar{v}^{\gamma}) \Bigg], \end{aligned} \tag{5.3}$$

where the new scalar functions $\hat{\phi}(x^1, x^2, t)$ and $\bar{\phi}(x^1, x^2, t)$ are the Lagrange multipliers for the kinematic boundary conditions on the surfaces $x^3 = \beta$ and $x^3 = \alpha$, respectively.

If we define a scalar function $\phi(x^1, x^2, t)$ as an arbitrary extension of $\hat{\phi}$ and $\bar{\phi}$ into the fluid domain, the integrals involving $\hat{\phi}$ and $\bar{\phi}$ in (5.3) can be written as

$$\iint \hat{\phi}\left(-\hat{v}^3 + \beta_{,\gamma}\hat{v}^\gamma\right)\,\mathrm{d}x^1\,\mathrm{d}x^2 - \iint \bar{\phi}\left(-\bar{v}^3 + \alpha_{,\gamma}\bar{v}^\gamma\right)\,\mathrm{d}x^1\,\mathrm{d}x^2$$
$$= -\int_{\partial\Omega} \phi\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S = -\int_{\Omega} \mathbf{v}\cdot\nabla\phi\,\mathrm{d}V, \tag{5.4}$$

and

$$\iint \left(\hat{\phi}\beta_{,t} - \bar{\phi}\alpha_{,t}\right)\,\mathrm{d}x^1\,\mathrm{d}x^2 = \iint \left[\frac{d}{dt}\int_\alpha^\beta \phi\,\mathrm{d}x^3 - \int_\alpha^\beta \phi_{,t}\,\mathrm{d}x^3\right]\,\mathrm{d}x^1\,\mathrm{d}x^2. \tag{5.5}$$

Substituting (5.4) and (5.5) in (5.3), we obtain another useful expression for the Lagrangian:

$$L = \iint \,\mathrm{d}x^1\,\mathrm{d}x^2\left[-\rho\int_\alpha^\beta \{\phi_{,t} + \mathbf{v}\cdot(\nabla\phi - \tfrac{1}{2}\mathbf{v})\}\,\mathrm{d}x^3 - \frac{\rho g}{2}\left(\beta^2 - \alpha^2\right) - \left(\hat{p}\beta - \bar{p}\alpha\right)\right], \tag{5.6}$$

where the first term on the right-hand side of (5.5) is dropped since it contributes only the end terms to the action integral $\int L\,\mathrm{d}t$, and thus it will not affect the equations (derived below) based on the principle of least action.

In taking variations on the action integral $\int L\,\mathrm{d}t$, we selectively use the expressions in (5.3) and (5.6) to simplify the derivation. Specifically, we use (5.3) to take variation of ϕ and (5.6) to take variation of α, β and $\mathbf{v}$. The corresponding variation of the action integral can be obtained following the scheme described in Luke [4]:

$$\begin{aligned}\delta\int L\,\mathrm{d}t = \;& \rho\iiint \,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}t\Big[\int_\alpha^\beta \delta\mathbf{v}\cdot(\mathbf{v} - \nabla\phi)\,\mathrm{d}x^3 + \\ & +\delta\hat{\phi}(\beta_{,t} - \hat{v}^3 + \beta_{,\gamma}\hat{v}^\gamma) - \delta\bar{\phi}(\alpha_{,t} - \bar{v}^3 + \alpha_{,\gamma}\bar{v}^\gamma) + \\ & +\tfrac{1}{\rho}\delta\beta\left(p^*|_{x^3=\beta} - \hat{p}\right) - \tfrac{1}{\rho}\delta\alpha\left(p^*|_{x^3=\alpha} - \bar{p}\right)\Big].\end{aligned} \tag{5.7}$$

Here the scalar function $p^* = p^*\left(x^1, x^2, x^3, t\right)$ is defined as

$$p^* \equiv -\rho\{\phi_{,t} + \mathbf{v}\cdot(\nabla\phi - \tfrac{1}{2}\mathbf{v}) + gx^3\}, \tag{5.8}$$

which can be written on the top and bottom boundaries as

$$\begin{aligned}p^* &= -\rho\left\{\hat{\phi}_{,t} + \hat{v}^\gamma\hat{\phi}_\gamma - \tfrac{1}{2}\mathbf{v}\cdot\mathbf{v} + g\beta\right\}, \quad x^3 = \beta, \\ p^* &= -\rho\left\{\bar{\phi}_{,t} + \bar{v}^\gamma\bar{\phi}_\gamma - \tfrac{1}{2}\mathbf{v}\cdot\mathbf{v} + g\alpha\right\}, \quad x^3 = \alpha.\end{aligned} \tag{5.9}$$

Since $\delta\mathbf{v}$, $\delta\phi$, $\delta\beta$ and $\delta\alpha$ are independent, (5.7) leads to the following equations:

$$\int_\Omega \delta\mathbf{v}\cdot(\mathbf{v} - \nabla\phi)\,\mathrm{d}V = 0, \tag{5.10}$$

$$\beta_{,t} - \hat{v}^3 + \beta_{,\gamma}\hat{v}^\gamma = 0, \quad \alpha_{,t} - \bar{v}^3 + \alpha_{,\gamma}\bar{v}^\gamma = 0, \tag{5.11}$$

$$\begin{aligned}\hat{\phi}_{,t} + \hat{v}^\gamma\hat{\phi}_\gamma - \tfrac{1}{2}\mathbf{v}\cdot\mathbf{v} + g\beta &= \frac{\hat{p}}{\rho}, \quad x^3 = \beta, \\ \bar{\phi}_{,t} + \bar{v}^\gamma\bar{\phi}_\gamma - \tfrac{1}{2}\mathbf{v}\cdot\mathbf{v} + g\alpha &= \frac{\bar{p}}{\rho}, \quad x^3 = \alpha.\end{aligned} \tag{5.12}$$

If we did not put any restriction on the variation $\delta\mathbf{v}$, (5.10) could be written as $\mathbf{v} = \nabla\phi$. But we assumed $\delta\mathbf{v}$ to be divergence-free. We shall discuss the implication of the weak formulation given in (5.10) when $\nabla \cdot \delta\mathbf{v} = 0$ later in Section 6. Before that, we shall first show how to derive the GN equations for irrotational flows using Hamilton's principle.

5.2. GN EQUATIONS FOR IRROTATIONAL FLOWS

Hamilton's principle leads to a new hierarchy of approximate equations if we use the same kinematic approximations used in obtaining the GN equations. The counterpart of the Level I GN equations has been derived by Miles and Salmon [18] and shown to be equivalent to the GN equations under certain conditions. We begin with the same problem to derive the new GN equations for non-uniform depth, which was not treated by Miles and Salmon [18] explicitly.

5.2.1. *Level IGN equations*

We use the same kinematic approximations given by (4.2) to obtain the Lagrangian in (5.3) as $L = \rho \iint \mathcal{L} \mathrm{d}x^1\, \mathrm{d}x^2$ where the Lagrangian density $\mathcal{L}$ is given as

$$\begin{aligned}\mathcal{L} = {} & \frac{\eta}{2}\left\{\mathbf{u}^2 + \tfrac{1}{3}(\eta\nabla\cdot\mathbf{u})^2 - \eta\nabla\cdot\mathbf{u}\mathcal{D}\alpha + (\mathcal{D}\alpha)^2\right\} \\ & + \hat{\phi}\left\{\mathcal{D}\eta + \eta\nabla\cdot\mathbf{u}\right\} \quad - \frac{g}{2}(\beta^2 - \alpha^2) - \frac{1}{\rho}\left(\hat{p}\beta - \bar{p}\alpha\right).\end{aligned} \tag{5.13}$$

From (5.13), the variation for $\mathbf{u}$ yields:

$$\nabla\hat{\phi} = \mathbf{u} + \frac{1}{3\eta}\nabla(\eta^2\mathcal{D}\eta) - \frac{1}{2\eta}\nabla(\eta^2\mathcal{D}\alpha) + \mathcal{D}\alpha\nabla\alpha + \frac{1}{2}\mathcal{D}\eta\nabla\alpha, \tag{5.14}$$

and the variation for β yields:

$$\hat{\phi}_{,t} + \mathbf{u}\cdot\nabla\hat{\phi} - \frac{1}{2}\mathbf{u}^2 - \frac{1}{2}(\mathcal{D}\beta)^2 + g\beta + \frac{\hat{p}}{\rho} = 0, \tag{5.15}$$

where (4.13) is used to eliminate $\nabla\cdot\mathbf{v}$. We can also derive (5.15) directly from (5.12). Note that here we have assumed $\alpha(x^1, x^2, t)$ is a given function. However, α can be treated also as a free surface, *e.g.*, in the water-fall problem. In such a case, one also needs to take the variation of the Lagrangian with respect to α to obtain the dynamic boundary condition on the bottom free surface.

As mentioned by Miles and Salmon [18], the equations given by (5.14) and (5.15) are equivalent to the Level I GN equations when the initial values of $\mathbf{u}$ and β are given such that the potential vorticity, Π, defined in their Equations (5.3) and (C 3b) vanishes. This requirement is equivalent to the compatibility condition for (5.14): the potential vorticity is identical to the curl of the right-hand side of (5.14), which should vanish to make $\hat{\phi}$ exist. There are two trivial cases in which the compatibility condition is satisfied. One is when the flow starts from rest and the other is when the flow is two dimensional, *i.e.*, $\mathbf{u} = \mathbf{u}(x^1, t)$. When the flow is two dimensional, we can obtain (4.15) from (5.15) after differentiating it with respect to x^1 and then eliminating $\hat{\phi}$ by using (5.14).

When the lower surface is flat at all times, *i.e.* , $\alpha = -h$, where h is the constant water depth, and the free-surface elevation is 'small', we can linearize (5.14) as

$$\nabla\hat{\phi} = \mathbf{u} + \frac{h}{3}\nabla\beta_t,$$

which implies that $\mathbf{u}$ is irrotational and therefore can be expressed in terms of a depth-integrated potential.

5.2.2. *Higher-level equations*

The general divergence-free velocity vector $\mathbf{v}(x^1, x^2, x^3, t)$ can be written by a vector potential $\boldsymbol{\Phi} = (\Phi^1, \Phi^2, 0)$:

$$v^\gamma = \Phi^\gamma_{,3}, \quad v^3 = -\Phi^\gamma_{,\gamma}, \tag{5.16}$$

We now use the same approximation given by (4.7):

$$\Phi^\gamma(x^1, x^2, t) = f_n(s)\Phi^\gamma_n(x^1, x^2, t), \quad n = 0, 1, \dots, K, \tag{5.17a}$$

$$v^\gamma = \frac{2}{\eta} f'_n(s)\Phi^\gamma_n, \tag{5.17b}$$

$$v^3 = -f_n(s)\Phi^\gamma_{n,\gamma} + \frac{1}{\eta} f'_n(s)\Phi^\gamma_n(2\zeta_{,\gamma} + s\eta_{,\gamma}). \tag{5.17c}$$

Substituting (5.17) in (5.3), we can obtain the Lagrangian density $\mathcal{L}$ as

$$\begin{aligned} \mathcal{L} = {} & \hat{\phi}\left(\beta_{,t} + f_n(1)\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_n\right) - \bar{\phi}\left(\alpha_{,t} + f_n(-1)\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_n\right) \\ & - \frac{\eta}{4} A_{mn}\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_m\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_n + \left(B^0_{mn}\nabla\zeta + \frac{1}{2}B^1_{mn}\nabla\eta\right)\cdot\boldsymbol{\Phi}_n\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_m \\ & - \frac{1}{\eta} C^0_{mn}\left(\boldsymbol{\Phi}_m\cdot\boldsymbol{\Phi}_n + \nabla\zeta\cdot\boldsymbol{\Phi}_m\nabla\zeta\cdot\boldsymbol{\Phi}_n\right) \\ & - \frac{1}{\eta} C^1_{mn}\nabla\zeta\cdot\boldsymbol{\Phi}_m\nabla\eta\cdot\boldsymbol{\Phi}_n - \frac{1}{4\eta} C^2_{mn}\nabla\eta\cdot\boldsymbol{\Phi}_m\nabla\eta\cdot\boldsymbol{\Phi}_n \\ & - \frac{g}{2}\left(\beta^2 - \alpha^2\right) - \frac{\hat{p}}{\rho}\beta + \frac{\bar{p}}{\rho}\alpha, \end{aligned} \tag{5.18}$$

where the coefficients A_{mn}, $B^{0,1}_{mn}$ and $C^{0,1}_{mn}$ are defined by

$$A_{mn} = \int_{-1}^{1} f_m f_n \, \mathrm{d}s, \quad B^k_{mn} = \int_{-1}^{1} s^k f'_m f_n \, \mathrm{d}s, \quad C^k_{mn} = \int_{-1}^{1} s^k f'_m f'_n \, \mathrm{d}s.$$

The stationary condition for the action integral $\iiint \mathcal{L}\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}t$ can be given by the Euler–Lagrange equations (see for example, Goldstein [26, Chapter 8]). The stationary condition for the variation of the Lagrangian, (5.18), with respect to β and α gives the following evolution equations for $\hat{\phi}$ and $\bar{\phi}$, respectively:

$$
\begin{aligned}
\hat{\phi}_{,t} \;&=\; -\frac{\partial \mathcal{L}}{\partial \beta} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial (\nabla \beta)} \\
&=\; -\tfrac{1}{4} A_{mn} \nabla \cdot \mathbf{\Phi}_m \nabla \cdot \mathbf{\Phi}_n - \tfrac{1}{2}\left(B^0_{mn} + B^1_{mn}\right) \nabla \cdot \left(\mathbf{\Phi}_n \nabla \cdot \mathbf{\Phi}_m\right) \\
&\quad + \frac{1}{\eta^2} C^0_{mn} \left(\mathbf{\Phi}_m \cdot \mathbf{\Phi}_n + \nabla \zeta \cdot \mathbf{\Phi}_m \nabla \zeta \cdot \mathbf{\Phi}_n\right) \\
&\quad + \frac{1}{\eta^2} C^1_{mn} \nabla \zeta \cdot \mathbf{\Phi}_m \nabla \eta \cdot \mathbf{\Phi}_n + \frac{1}{4\eta^2} C^2_{mn} \nabla \eta \cdot \mathbf{\Phi}_m \nabla \eta \cdot \mathbf{\Phi}_n \\
&\quad + \frac{1}{\eta} C^0_{mn} \nabla \cdot \left(\mathbf{\Phi}_m \nabla \zeta \cdot \mathbf{\Phi}_n\right) + \frac{1}{\eta} C^1_{mn} \nabla \cdot \left\{\mathbf{\Phi}_m \left(\nabla \zeta + \tfrac{1}{2} \nabla \eta\right) \cdot \mathbf{\Phi}_n\right\} \\
&\quad - \frac{1}{2\eta} C^2_{mn} \nabla \cdot \left(\mathbf{\Phi}_m \nabla \eta \cdot \mathbf{\Phi}_n\right) - g\beta - \hat{p}/\rho,
\end{aligned}
\tag{5.19}
$$

$$
\begin{aligned}
\overline{\phi}_{,t} &= \frac{\partial \mathcal{L}}{\partial \alpha} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial (\nabla \alpha)} \\
&= -\tfrac{1}{4} A_{mn} \nabla \cdot \mathbf{\Phi}_m \nabla \cdot \mathbf{\Phi}_n - \tfrac{1}{2}\left(B^0_{mn} - B^1_{mn}\right) \nabla \cdot \left(\mathbf{\Phi}_n \nabla \cdot \mathbf{\Phi}_m\right) \\
&\quad + \frac{1}{\eta^2} C^0_{mn} \left(\mathbf{\Phi}_m \cdot \mathbf{\Phi}_n + \nabla \zeta \cdot \mathbf{\Phi}_m \nabla \zeta \cdot \mathbf{\Phi}_n\right) \\
&\quad + \frac{1}{\eta^2} C^1_{mn} \nabla \zeta \cdot \mathbf{\Phi}_m \nabla \eta \cdot \mathbf{\Phi}_n + \frac{1}{4\eta^2} C^2_{mn} \nabla \eta \cdot \mathbf{\Phi}_m \nabla \eta \cdot \mathbf{\Phi}_n \\
&\quad - \frac{1}{\eta} C^0_{mn} \nabla \cdot \left(\mathbf{\Phi}_m \nabla \zeta \cdot \mathbf{\Phi}_n\right) + \frac{1}{\eta} C^1_{mn} \nabla \cdot \left\{\mathbf{\Phi}_m \left(\nabla \zeta - \tfrac{1}{2} \nabla \eta\right) \cdot \mathbf{\Phi}_n\right\} \\
&\quad - \frac{1}{2\eta} C^2_{mn} \nabla \cdot \left(\mathbf{\Phi}_m \nabla \eta \cdot \mathbf{\Phi}_n\right) - g\alpha - \overline{p}/\rho,
\end{aligned}
\tag{5.20}
$$

The stationary condition for the variation of $\hat{\phi}$ and $\bar{\phi}$ gives the kinematic boundary conditions:

$$
\beta_{,t} = -f_n(1) \nabla \cdot \mathbf{\Phi}_n, \qquad \alpha_{,t} = -f_n(-1) \nabla \cdot \mathbf{\Phi}_n. \tag{5.21}
$$

Note that if we particularly choose f_n's as the same polynomials used in Section 4.4, we can show that (5.21) is equivalent to the statements of mass conservation (see (4.19)).

And finally, for the variation of $\mathbf{\Phi}_m (m = 0, 1, \ldots, K)$, we obtain a system of coupled second-order partial differential equations to be solved for $\mathbf{\Phi}_m$:

$$
\begin{aligned}
&\tfrac{1}{2}A_{mn}\nabla(\eta\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_n)+(B^0_{nm}\nabla\zeta+\tfrac{1}{2}B^1_{nm}\nabla\eta)\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_n\\
&-\nabla\left\{\left(B^0_{mn}\nabla\zeta+\tfrac{1}{2}B^1_{mn}\nabla\eta\right)\cdot\boldsymbol{\Phi}_n\right\}-\frac{2}{\eta}C^0_{mn}\{\boldsymbol{\Phi}_n+\nabla\zeta(\nabla\zeta\cdot\boldsymbol{\Phi}_n)\}\\
&-\frac{1}{\eta}C^1_{mn}\left\{\nabla\zeta\,(\nabla\eta\cdot\boldsymbol{\Phi}_n)+\nabla\eta\,(\nabla\zeta\cdot\boldsymbol{\Phi}_n)\right\}-\frac{1}{2\eta}C^2_{mn}\nabla\eta\,(\nabla\eta\cdot\boldsymbol{\Phi}_n)\\
&=f_m(1)\nabla\hat{\phi}-f_m(-1)\nabla\bar{\phi},\quad m=0,1,\ldots,K.
\end{aligned}
\tag{5.22}
$$

The approximate set of equations given by (5.19)–(5.22), which we shall refer to as the IGN equations hereafter, has a considerably simpler structure compared with the set of the GN equations given by (4.19) and (4.20). The order of the differential equations is reduced from three to two. The number of operations is also reduced from $O(K^3)$ to $O(K^2)$.

Miles and Salmon [18] already showed that the equations, termed the IGN equations here, are equivalent to the GN equations at the first level of approximation. Naturally, one may ask whether the two equations are equivalent at the higher-level approximation. If not, one may also ask whether the solutions of the two different sets of equations converge to the same solutions. These will be discussed next.

6. Consequences of the variational principles

We derived the variational principles from the weak formulation of Euler's equations given by (2.4):

$$\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}=-\nabla\left(p+\rho g x^3\right). \tag{6.1}$$

In the context of the principle of virtual work, the above equation is weakly satisfied as

$$\int_\Omega \delta\mathbf{X}\cdot\left\{\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}+\nabla\left(p+\rho g x^3\right)\right\}\,\mathrm{d}V=0. \tag{6.2}$$

And as shown in Section 5.1, Hamilton's principle applied to the same problem leads to the following weak formulation:

$$\int_\Omega \delta\mathbf{X}\cdot(\mathbf{v}-\nabla\phi)\,\mathrm{d}V=0, \tag{6.3}$$

$$p^*=\hat{p},x^3=\beta, \tag{6.4a}$$

$$p^*=\bar{p},x^3=\alpha. \tag{6.4b}$$

where the scalar function $p^*=p^*\left(x^1,x^2,x^3,t\right)$ is defined by (5.8).

If the virtual displacements (or the test functions) $\delta\mathbf{X}$ were chosen arbitrarily from a complete set of functions, it is clear that (6.2) is equivalent to (6.1). We can also show that the solution of (6.3) satisfies (6.1) since the scalar function ϕ can be identified as the velocity potential and (5.8) as Euler's integral. However, in the variational principles presented here, the test functions $\delta\mathbf{X}$ were chosen among the divergence-free functions. One may question whether the solutions of the variational problems still satisfy Euler's equations with that restriction on the test functions. We shall answer this question next.

6.1. Implication of the weak formulation

If we satisfy a vector equation $\mathbf{r} = 0$ in weak sense by a test function $\delta\mathbf{X} = \nabla \times \delta\mathbf{\Psi}$ we have

$$\begin{aligned} \int_\Omega \delta\mathbf{X} \cdot \mathbf{r}\, \mathrm{d}V &= \int_\Omega (\nabla \times \delta\mathbf{\Psi}) \cdot \mathbf{r}\, \mathrm{d}V \\ &= \int_\Omega \delta\mathbf{\Psi} \cdot (\nabla \times \mathbf{r})\ \mathrm{d}V - \int_{\partial\Omega} \delta\mathbf{\Psi} \cdot (\mathbf{n} \times \mathbf{r})\ \mathrm{d}S = 0, \end{aligned} \tag{6.5}$$

where the divergence theorem has been applied. Since the horizontal components of $\delta\mathbf{\Psi}$ are arbitrary, $\int_\Omega \delta\mathbf{X} \cdot \mathbf{r}\, \mathrm{d}V = 0$ implies

$$\mathbf{e}_\gamma \cdot (\nabla \times \mathbf{r}) = 0 \quad \text{in} \quad \Omega, \tag{6.6}$$

$$\mathbf{s}_\gamma \cdot \mathbf{r} = 0 \quad \text{on} \quad \partial\Omega. \tag{6.7}$$

Here the vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ are defined as the tangential vectors on the boundaries whose horizontal components are parallel to the x^1 and x^2 axes, respectively, *i.e.*,

$$\mathbf{s}_1 = -\mathbf{n} \times \mathbf{e}_2, \qquad \mathbf{s}_2 = \mathbf{n} \times \mathbf{e}_1.$$

Note that only the tangential components of the original equation $\mathbf{r} = 0$ are satisfied on the boundary. In the fluid domain, the horizontal components of $\nabla \times \mathbf{r}$ are equal to zero.

We can also show that the vertical component of $\nabla \times \mathbf{r}$ is equal to zero. Consider a volume inside a cylindrical surface which consists of an arbitrary horizontal surface S_Z, the vertical surface S_W and the bottom surface S_B which is part of the surface $x^3 = \alpha$. Since $\nabla \times \mathbf{r}$ is divergence-free, we have

$$\int_{S_Z} (\nabla \times \mathbf{r}) \cdot \mathbf{e}_3\, \mathrm{d}S = -\int_{S_W} (\nabla \times \mathbf{r}) \cdot \mathbf{n}\, \mathrm{d}S - \int_{S_B} (\nabla \times \mathbf{r}) \cdot \mathbf{n}\, \mathrm{d}S.$$

The surface integral over S_W vanishes from (6.6) since the normal vectors on the surface S_W are horizontal. Moreover, using the Stokes theorem, we have

$$\begin{aligned} \int_{S_B} (\nabla \times \mathbf{r}) \cdot \mathbf{n}\, \mathrm{d}S &= \oint_{\partial S_B} \mathbf{r} \cdot\ \mathrm{d}\mathbf{x} = \oint_{\partial S_B} \left(r^\gamma + \alpha_{,\gamma} r^3\right)\ \mathrm{d}x^\gamma \\ = -\oint_{\partial S_B} \sqrt{1 + \alpha_{,1}^2 + \alpha_{,2}^2}\, \mathbf{r} \cdot \mathbf{s}_\gamma \mathrm{d}x^\gamma &= 0 \end{aligned}$$

from (6.7). Then the surface integral over S_Z is equal to zero and so is $(\nabla \times \mathbf{r}) \cdot \mathbf{e}_3$ since the surface S_Z is arbitrary in the fluid domain. As a result, we have

$$\nabla \times \mathbf{r} = 0 \quad \text{in} \quad \Omega, \tag{6.8}$$

rather than $\mathbf{r} = 0$ in the fluid domain.

Now we are ready to investigate the consequences of the weak formulations (6.2) and (6.3). We begin with the 'exact problem' where no approximations are made on the velocity fields. After that, we shall consider the 'discrete problem' where the velocity profiles are approximated by a finite number of interpolation functions in the vertical direction.

6.1.1. *Exact problem*

If we apply the above result to the principle of virtual work, with $\mathbf{r} = \rho(D\mathbf{v}/Dt) + \nabla p + \rho g \mathbf{e}_3$, we have

$$\nabla \times \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = 0 \quad \text{in} \quad \Omega \tag{6.9a}$$

$$\mathbf{s}_\gamma \cdot \left(\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} + \frac{1}{\rho}\nabla p + g\mathbf{e}_3 \right) = 0 \quad \text{on} \quad \partial\Omega, \tag{6.9b}$$

the latter being the vorticity transport equation in the fluid domain with the tangential components of Euler's equations satisfied on the boundaries. On the other hand, Hamilton's principle, with $\mathbf{r} = \mathbf{v} - \nabla\phi$, gives

$$\nabla \times \mathbf{v} = 0 \quad \text{in} \quad \Omega, \tag{6.10a}$$

$$\mathbf{s}_\gamma \cdot (\mathbf{v} - \nabla\phi) = 0 \quad \text{on} \quad \partial\Omega. \tag{6.10b}$$

We have the irrotationality condition as the governing equation in the fluid domain, and the tangential velocity components are given as the tangential gradients of ϕ. Note that the value of ϕ in the fluid domain does not affect the solution $\mathbf{v}$ in (6.10). Only the values on the top and bottom surfaces contribute to the solution. If we specify ϕ as the solution of a Dirichlet problem

$$\nabla^2\phi = 0 \quad \text{in} \quad \Omega, \tag{6.11a}$$

$$\phi|_{x^3=\beta} = \hat{\phi}, \quad \phi|_{x^3=\alpha} = \bar{\phi}, \tag{6.11b}$$

and identify ϕ as the velocity potential, (5.8) can be interpreted as Euler's integral and the scalar function p^* as the pressure in the fluid domain.

Note that the solution of Hamilton's principle leads only to irrotational flows, whereas the GN equations permit rotational flows. We can conclude then that the solutions of the principle of virtual work and Hamilton's principle converge to the same solution if the flow starts from rest.

6.1.2. *Discrete problem*

Let us consider (6.5) again, and write it as

$$\iint \mathrm{d}x^1\,\mathrm{d}x^2 \left[\int_\alpha^\beta \delta\mathbf{\Psi} \cdot (\nabla \times \mathbf{r})\;\mathrm{d}x^3 - \delta\hat{\mathbf{\Psi}} \cdot \left(\hat{\boldsymbol{\nu}} \times \mathbf{r}\right) + \delta\bar{\mathbf{\Psi}} \cdot (\bar{\boldsymbol{\nu}} \times \mathbf{r}) \right] = 0, \tag{6.12}$$

where the vectors $\hat{\boldsymbol{\nu}} = \left(\hat{\nu}^1, \hat{\nu}^2, \hat{\nu}^3\right)$ and $\bar{\boldsymbol{\nu}} = \left(\bar{\nu}^1, \bar{\nu}^2, \bar{\nu}^3\right)$ are defined as $\hat{\nu}^\gamma = -\beta_{,\gamma}$, $\hat{\nu}^3 = 1$ and $\bar{\nu}^\gamma = -\alpha_{,\gamma}$, $\bar{\nu}^3 = 1$.

By substituting $\delta\mathbf{\Psi}(x^1, x^2, x^3, t) = f_n(s)\delta\Psi_n(x^1, x^2, x^3, t)$ in (6.12), we can obtain

$$\iint \delta\Psi_n \cdot \left[\frac{\eta}{2} \int_{-1}^{1} f_n\,(s)\,\nabla \times \mathbf{r}\;\mathrm{d}s - f_n(1)\hat{\boldsymbol{\nu}} \times \hat{\mathbf{r}} + f_n(-1)\bar{\boldsymbol{\nu}} \times \bar{\mathbf{r}} \right] \mathrm{d}x^1\,\mathrm{d}x^2 = 0.$$

Since $\delta\Psi_n$ are arbitrary, we must have

$$\begin{aligned} \mathbf{e}_\gamma \cdot \Bigg[&\frac{\eta}{2} \int_{-1}^{1} f_n\,(s)\,\nabla \times \mathbf{r}\;\mathrm{d}s - \\ &- f_n(1)\hat{\boldsymbol{\nu}} \times \hat{\mathbf{r}} + f_n(-1)\bar{\boldsymbol{\nu}} \times \bar{\mathbf{r}} \Bigg] = 0, \quad n = 0, 1, \ldots, K. \end{aligned} \tag{6.13}$$

For the particular test functions $f_n(s)$, which vanish at $s = 1$ and -1, *i.e.*,

$$f_n(-1) = f_n(1) = 0, \tag{6.14}$$

we have, from (6.13),

$$\mathbf{e}_\gamma \cdot \int_{-1}^{1} f_n(s)\, (\nabla \times \mathbf{r})\; \mathrm{d}s = 0, \tag{6.15}$$

which states that the horizontal components of $\nabla \times \mathbf{r}$ are weakly zero in the fluid domain.

If we apply (6.15) to (6.2) and (6.3), we have

$$\mathbf{e}_\gamma \cdot \int_{-1}^{1} f_n(s) \nabla \times \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\, \mathrm{d}s = 0 \tag{6.16}$$

and

$$\mathbf{e}_\gamma \cdot \int_{-1}^{1} f_n(s) \nabla \times \mathbf{v}\, \mathrm{d}s = 0, \tag{6.17}$$

for the solution of the GN equations and IGN equations, respectively. In the IGN equations, the vorticity is minimized in the fluid domain whereas in the GN equations the curl of acceleration is minimized. This suggests that the two depth-integrated equations do not necessarily give the same results, although they converge to the same solution when K is sufficiently large as we shall argue next.

Taking the time derivative of the IGN equations given in (6.17), we have

$$\mathbf{e}_\gamma \cdot \left[\frac{\eta}{2} \int_{-1}^{1} f_n(s) \nabla \times \mathbf{v}_{,t}\; \mathrm{d}s \right] = 0.$$

And adding an irrotational term, $\frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}) = (\mathbf{v}\cdot\nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{v})$, to $\mathbf{v}_{,t}$, we have

$$\mathbf{e}_\gamma \cdot \left[\frac{\eta}{2} \int_{-1}^{1} f_n(s) \nabla \times \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\; \mathrm{d}s \right] = \mathbf{e}_\gamma \cdot \left[\frac{\eta}{2} \int_{-1}^{1} f_n(s) \nabla \times (\mathbf{v} \times \nabla \times \mathbf{v})\; \mathrm{d}s \right]. \tag{6.18}$$

The right-hand side of (6.18) can be interpreted as the residuals of the GN equations when we substitute the solution of the IGN equations in the GN equations. The residual is linearly dependent on the vorticity which is minimized in the IGN equations and approaches to zero as the level of approximation increases. From these results, we can deduce that the solutions of the two approximate equations, the GN and the IGN equations, converge to the same exact solution, if it exists and if the fluid starts from rest, although they may not give the same solution for a particular, finite Level K[1].

6.2. Conservation laws

Shields and Webster [19] showed that the GN equations preserve the conservation laws of the physical model, such as the momentum, mechanical energy, and circulation. Miles and Salmon [18] showed that the conservation laws can be inferred directly from the symmetries of the Lagrangian of the Level I GN equations, and they described the conservation of circulation alternatively as the conservation of potential vorticity. For the IGN equations, the same conservation laws can be derived from the properties of the variational principle since the Hamiltonian structure is preserved in the discrete system.

The conservation of energy comes from the conservation of the Hamiltonian. The Lagrangian L can be written as

[1] The authors are indebted to Prof. John V. Wehausen of U.C. Berkeley for bringing this point to their attention.

$$L = \rho \iint \left(\hat{\phi} \beta_{,t} - \bar{\phi} \alpha_{,t} \right) \, \mathrm{d}x^1 \, \mathrm{d}x^2 - H, \tag{6.19}$$

where the Hamiltonian H is defined by

$$H = \rho \int_\Omega \mathbf{v} \cdot \left(\nabla\phi - \frac{1}{2}\mathbf{v} \right) \, \mathrm{d}V + \rho g \int_\Omega x^3 \, \mathrm{d}V + \iint \left(\hat{p}\beta - \bar{p}\alpha \right) \, \mathrm{d}x^1 \, \mathrm{d}x^2. \tag{6.20}$$

If $\mathbf{v}$ satisfies the variational equation, the first term of the Hamiltonian can be identified as the kinetic energy because the relation $\int_\Omega \mathbf{v} \cdot (\mathbf{v} - \nabla\phi) \, \mathrm{d}V = 0$ can be obtained from (6.3) by substituting $\delta\mathbf{X} = \mathbf{v}$. The sum of the volume integrals in (6.20) is the total mechanical energy of the fluid and will be denoted by E. From the well-known Hamiltonian conservation law (see *e.g.*, Goldstein [26, Chapter 8]).

$$\frac{\mathrm{d}}{\mathrm{dt}} H[\mathbf{v}, \phi, \alpha, \beta, t] = \frac{\partial}{\partial t} H[\mathbf{v}, \phi, \alpha, \beta, t],$$

and therefore, we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t} \iint \left(\hat{p}\beta - \bar{p}\alpha \right) \, \mathrm{d}x^1 \, \mathrm{d}x^2 = \iint \left(\hat{p}_{,t}\beta - \bar{p}_{,t}\alpha \right) \, \mathrm{d}x^1 \, \mathrm{d}x^2 \tag{6.21}$$

or

$$\frac{\mathrm{d}E}{\mathrm{d}t} = - \iint \left(\hat{p}\beta_{,t} - \bar{p}\alpha_{,t} \right) \mathrm{d}x^1 \, \mathrm{d}x^2 = - \int_{\partial\Omega} p\mathbf{v} \cdot \mathbf{n} \mathrm{d}S, \tag{6.22}$$

which is the statement of conservation of energy.

Since the solutions of the IGN equations satisfy the conservation of mechanical energy, the conservation of momentum comes from the fact that the variational problem is independent of the rigid-body translation (Green and Naghdi [10]). If we have a solution set $\{\mathbf{v}, \alpha, \beta\}$, we need to show that $\left\{\mathbf{v} + \mathbf{U}, \alpha + U^3 t, \beta + U^3 t\right\}$ is also a solution set, which can easily be shown by substituting it in (6.3) and (6.4) with ϕ replaced by

$$\phi \to \phi + \mathbf{U} \cdot \mathbf{x} - \frac{1}{2} |\mathbf{U}|^2 \, t, \tag{6.23}$$

where $\mathbf{U} = \left(U^1, U^2, U^3\right)$ is an arbitrary constant vector. An interesting consequence of the above conservation laws is that, at the lowest level of approximation, $K = 1$, the two variational principles, namely the principle of virtual work and Hamilton's principle, lead to the same solution since the GN equations can be derived solely from the conservation laws as was shown by Green and Naghdi [10].

The conservation of circulation is realized in a slightly different way in the GN and IGN equations. The conservation of cross-sheet and in-sheet circulations in the GN equations, which have been previously shown by Shields and Webster [19], can be derived from [6.13) by substituting $\mathbf{r} = (\mathrm{D}\mathbf{v}/\mathrm{D}t)$ as shown in Appendix A. In the IGN equations, the conservation of the in-sheet and cross-sheet circulations are satisfied more strongly than they are in the original GN equations because in the IGN equations, the moments of the in-sheet and cross-sheet circulations are zero, whereas in the GN equations the moments of the rate of change of circulations following the fluid particles, are zero. This can be deduced from (6.13) with $\mathbf{r} = \mathbf{v}$.

7. Concluding remarks

The GN equations are re-derived through the principle of virtual work and the divergence-free virtual displacements and by satisfying the momentum equations weakly. As a result, the integrated internal pressure variables are eliminated from the GN equations at any Level K of the approximation.

The GN equations for irrotational flows are derived at any Level K by use of a restricted Hamilton's principle. The new set of equations, called the IGN equations, has a considerably simpler structure compared with the original GN equations. In the GN equations given by (4.19) and (4.20), the number of unknowns are $2K+4$, including $2K+2$ vector potentials $\mathbf{\Phi}_n$ and α and β, whereas in the IGN equations given by (5.19) through (5.22), there are $2K+6$ unknowns, $\hat{\phi}$ and $\bar{\phi}$ being the additional unknowns. However, the order of the differential equations is reduced from 3 to 2 and the order of operations is reduced from $O(K^3)$ in the GN equations to $O(K^2)$ in the IGN equations. Moreover, it is found in Section 5.2.1 that the Level I GN and IGN equations are identical in two dimensions, and they provide the same solution in three dimensions if the potential vorticity is initially zero.

It is also shown that the solutions of the IGN equations converge to the same solutions of the GN equations as the level of approximation increases. On the other hand, the conservation of the in-sheet and cross-sheet circulations are satisfied more strongly by the IGN equations compared with the GN equations because the K moments of the in-sheet circulation and $K-1$ weighed moments of the cross-sheet circulation are identically zero in the solutions of the IGN equations.

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Appendix

WEAK CONSERVATION OF CIRCULATION

Consider a contour C_0 of length l lying in the (x^1, x^2) plane as shown in Figure 2. The contour is either closed as in Figure 2a or open as in Figure 2b. We define a length coordinate, λ, along this contour with $\lambda = 0$ and $\lambda = l$ corresponding to the end points of the contour. If the contour is closed, $\lambda = 0$ and $\lambda = l$ represent the same point. With C_0 as a generating curve, we can define a separate contour, $C(s, t)$, in the fluid sheet for a given value of the nondimensional vertical coordinate s. Then $C(s, t)$ can be expressed parametrically by $x^1(\lambda)$, $x^2(\lambda)$ and $x^3(\lambda, s, t) \equiv \zeta(\lambda, t) + s\frac{\eta(\lambda,t)}{2}$.

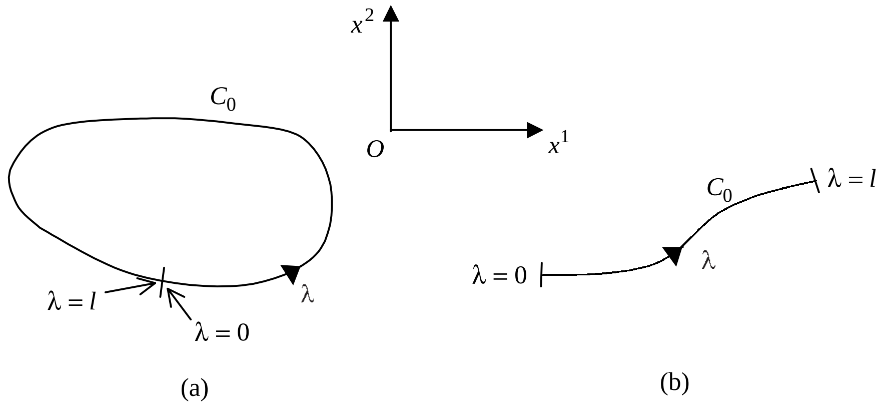

Figure 2. Definition of the contour C_0.

We define $\Gamma_{\mathbf{r}}(s)$ as a line integral of a vector $\mathbf{r} = (r^1, r^2, r^3)$ along the contour $C(s, t)$:

$$\Gamma_{\mathbf{r}}(s, t) \equiv \int_{C(s,t)} \mathbf{r} \cdot \mathrm{d}\mathbf{x}. \tag{A.1}$$

From Stokes' theorem on the vertical surface between $C(s, t)$ and C_0, we have

$$-\int_{C_0} \int_{-1}^{s} \frac{\eta}{2} \mathbf{n}_C \cdot (\nabla \times \mathbf{r})\ \mathrm{d}s\, \mathrm{d}\lambda = \Gamma_{\mathbf{r}}(s, t) - \Gamma_{\mathbf{r}}(-1, t) - \left[\int_{-1}^{s} \frac{\eta}{2} r^3\, \mathrm{d}s\right]_{\lambda=0}^{\lambda=l}. \tag{A.2}$$

We differentiate (A.2) to obtain

$$\gamma_{\mathbf{r}}(x, t) \equiv \Gamma'_{\mathbf{r}}(s, t) - \left[\frac{\eta}{2} r^3\right]_{\lambda=0}^{\lambda=l} = -\int_{C(s,t)} \frac{\eta}{2} \mathbf{n}_C \cdot (\nabla \times \mathbf{r})\ \mathrm{d}\lambda. \tag{A.3}$$

Multiplying (A.3) by $f_n(s)$; then integrating from $s = -1$ to 1, we obtain

$$-\int_{-1}^{1} f_n(s) \gamma_{\mathbf{r}}(s, t)\, \mathrm{d}s = \int_{C_0} \int_{-1}^{1} \frac{\eta}{2} f_n(s) \mathbf{n}_c \cdot \nabla \times \mathbf{r}\, \mathrm{d}s\, \mathrm{d}\lambda. \tag{A.4}$$

When C_0 is a closed circuit as in Figure 2a, $\gamma_{\mathbf{r}}(s, t) = \Gamma'_{\mathbf{r}}(s, t)$ and the left-hand side of (A.4) can be integrated by parts:

$$\begin{aligned} &\int_{-1}^{1} f'_n(s) \Gamma_{\mathbf{r}}(s, t)\, \mathrm{d}s - f_n(1) \Gamma_{\mathbf{r}}(1, t) + f_n(-1) \Gamma_{\mathbf{r}}(-1, t) \\ &= \int_{C_0} \int_{-1}^{1} \frac{\eta}{2} f_n(s) \mathbf{n}_c \cdot \nabla \times \mathbf{r}\, \mathrm{d}s\, \mathrm{d}\lambda. \end{aligned} \tag{A.5}$$

On the other hand, integrating (6.13) along the contour C_0, we obtain

$$\int_{C_0} \int_{-1}^{1} s \frac{\eta}{2} f_n(s) \mathbf{n}_c \cdot \nabla \times \mathbf{r}\, \mathrm{d}s\, \mathrm{d}\lambda = -f_n(1) \Gamma_{\mathbf{r}}(1, t) + f_n(-1) \Gamma_{\mathbf{r}}(-1, t). \tag{A.6}$$

From (A.4) and (A.6), we then have

$$\int_{-1}^{1} f_n(s) \gamma_{\mathbf{r}}(s, t)\, \mathrm{d}s = 0 \quad \text{if} \quad f_n(1) = f_n(-1) = 0 \tag{A.7}$$

for any circuit C_0. And from (A.5) and (A.6), we have

$$\int_{-1}^{1} f_n'(s)\Gamma_{bfr}(s,t)\,\mathrm{d}s = 0 \tag{A.8}$$

when C_0 is a closed circuit. When a polynomial function is chosen as the vertical interpolation function, *i.e.*, $f_n(s) = s^n$, $n = 0, 1, \ldots, K$, we have

$$\int_{-1}^{1} s^n \Gamma_{\mathbf{r}}(s,t)\,\mathrm{d}s = 0, n = 0, 1, \ldots, K-1, \tag{A.9}$$

$$\int_{-1}^{1} s^n (1-s^2)\gamma_{\mathbf{r}}(s,t)\,\mathrm{d}s = 0, n = 0, 1, \ldots, K-2. \tag{A.10}$$

The above results in (A.9) and (A.10) can be applied to the GN equations and IGN equations by replacing $\mathbf{r}$ by $\rho\dfrac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} + \nabla(p + gx^3)$ and $\mathbf{v} - \nabla\phi$, respectively. However, $\Gamma_{\mathbf{r}}(s,t)$ and $\gamma_{\mathbf{r}}(s,t)$ do not change if we add an arbitrary irrotational term to $\mathbf{r}$, which can be shown from (A.1) and (A.3). Therefore, we can replace $\mathbf{r}$ by $\dfrac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}$ and $\mathbf{v}$ when we apply (A.9) and (A.10) to the GN equations and IGN equations, respectively. When we substitute $\dfrac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}$, (A.9) and (A.10)) become equivalent to (4.4) and (4.10) of Shields and Webster [19], which are the statements of weak conservation of in-sheet and cross-sheet circulations, respectively. Specifically, the K moments of the in-sheet circulation and $K-1$ weighted moments of the cross-sheet circulation are conserved in the Kth Level GN equations. In the IGN equations, where we can substitute $\mathbf{v}$ for $\mathbf{r}$, the statement is more strongly given as the K moments of the in-sheet circulation and the $K-1$ weighted moments of the cross-sheet circulation are zero.

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